

# Value-at-Risk for Fixed Income portfolios –A comparison of alternative models

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## **Abstract**

The paper presents a case for a new method for computing the VaR for a set of fixed income securities based on extreme value theory that models the tail probabilities directly without making any assumption about the distribution of entire return process. It compares the estimates of VaR for a portfolio of fixed income securities across three methods: Variance-Covariance method, Historical Simulation method and Extreme Value method and finds that extreme value method provides the accurate VaR estimator in terms of correct failure ratio and the size of VaR.

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# 1 Introduction

“..the biggest problem we now have with the whole evolution of the risk is the fat-tailed problem, which is really creating very large conceptual difficulties. Because as we all know, the assumption of normality enables us to drop off the huge amount of complexity in our equations.. Because once you start putting in non-normality assumptions, which is unfortunately what characterizes the real world, then these issues become extremely difficult.”

— Alan Greenspan (1997).

Value-at-Risk (VaR) has been widely promoted by regulatory groups and embraced by financial institutions as a way of monitoring and managing market risk - the risk of loss due to adverse movements in interest rate, exchange rate, equity and commodity exposures - and as a basis for setting regulatory minimum capital standards. The revised Basle Accord, implemented in January 1998, allows banks to use VaR as a basis for determining how much additional capital must be set aside to cover market risk beyond that required for credit risk. Market related risk has become more relevant and important due to the trading activities and market positions taken by large banks. Another impetus for such a measure has come from the numerous and substantial losses that have arisen due to shortcomings in risk management procedures that failed to detect errors in derivatives pricing (Natwest, UBS), excessive risk taking (Orange County, Proctor and Gamble), as well as fraudulent behavior (Barings and Sumitomo).

VaR models usually use historical data to evaluate maximum (worst case) trading losses for a given portfolio over a certain holding period at a given confidence interval <sup>1</sup>.As a result, it is essentially determined by two parameters, (holding) time period and a confidence level. It is a measure of the loss (expressed in say rupees crores) on the portfolio that will not be exceeded by the end of the time period with the specified confidence level. If  $\alpha$  is the confidence level and N days is the time period, the calculation

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<sup>1</sup>To be more precise, VaR is the minimum, and not the maximum, loss that could be incurred in holding a portfolio in 1 out of 100 occurrences

of VaR is based on the probability distribution of changes in the portfolio value over N days. Specifically, VaR is set equal to the loss in the portfolio at the  $(1-\alpha)*100$  percentile point of the distribution.

A first important observation that VaR applies to the extreme lower tail of the return distribution, i.e. large losses far way from the mean. Bank regulators have recognized this and typically chosen  $\alpha$  equal to 99 percent. This number obviously reflects regulators' natural tendency for conservativeness in their prudential supervision of banks. The same tendency also comes out in Basle regulators' choice of holding period, the second important model parameter. While the industry typically uses daily VaRs for its internal risk control (Danielsson, Hartman and de Vries (2000)), for the purpose of determining their minimum regulatory capital against market risk, banks will be obliged to assume that they can not liquidate their trading portfolios quicker than within 10 business days. In order to facilitate the transition from their internal daily VaR models to the regulatory 10-day VaR models, the application of a simple square-root-of-time rule permitted. 10-day VaR, therefore, is derived by multiplying a 1-day VaR with a factor equal to square root of holding period. For the purpose of setting adequate capital, Basle accord suggests another factor by which the computed VaR for the 10-day period should be multiplied with.

While the VaR measure has been rightly criticised by the risk managers for being inadequate, it bridges the gap between the need to measure the risk accurately and for non-technical parties to be able to understand such a measure<sup>2</sup>. The concept of VaR has several shortcomings. First, it is only the minimum amount of losses (best of the worst scenarios) that can not indicate anything about the intensity of losses. For example, if VaR of a position is estimated to be Rs.100, we do not know whetehr the maximum loss is

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<sup>2</sup>It must be noted that the 'greeks' that purport to measure the sensitivity of a position with respect to changes in the underlying risk factor, while being simple, are not inofrmative about the probability of observing a large chnage in the underlying risk facor. VaR contains these two measures in a single number.

Rs.150 or Rs.1000. Other measures such as Expected Shortfall (ES) or Tail Conditional Expectation (TCE) are proposed to deal with nature of losses beyond the threshold defined by VaR (Danielsson (2000) and Acerbi(2000)). Second criticism against VaR has been that it, being linked to the extreme quantiles of the distribution of returns, is excessively volatile leading to difficulties in implimentation of capital norms. While this may not be such a serious issue from the point of regulatory capital, as the latter includes provision for factors other than market risk and typically larger by an order of magnitude than market risk VaR, it is a matter of concern for internal risk management purposes. For example, when VaR is used to allocate position limits to individual traders, high volatility of risk measures is serious problem as it is very hard to manage individual positions with highly volatile position limits.

Another property of VaR that is not typically recognised but has serious consequences for risk management is that it is not, in general, sub-additive in the sense that the sum of VaRs of components of a portfolio is not always greater than the VaR of a portfolio. This property would imply that a user has to recompute the entire portfolio VaR every time a new instrument enters or leaves the portfolio. The current practice of computing VaR for equity, interest and forex VaRs separately and summing up to arrive at the total portfolio VaR may, therefore, lead to misleading estimates of risk<sup>3</sup>. If a risk measure is sub-additive then computation by parts would always lead to a conservative estimate of the portfolio risk and if one is willing to be conservative such a measure risk would significantly reduce the costs of computation<sup>4</sup>. Notwithstanding these limitations, we, for the purpose of this paper, consider VaR as the measure of risk and concentrate on alternative

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<sup>3</sup>see Altzner, Delbaen, Eber and Heath (1998) and Acerbi (2000) for more details on this point.

<sup>4</sup>For a large clearing corporation or exchange that sets margins for brokers based on the riskiness of their portfolio, computational costs may not be a serious constraint. For a bank, that needs to evaluate the risk of its portfolio holdings on a frequent basis, on the otherhand, this could become an important constraint.

estimation methods.

Fundamentally, all the statistical risk modelling techniques fall into one of the three categories or a combination thereof. Fully parametric methods based on modelling the entire distribution of returns, usually with some form of conditional volatility, the non-parametric method of historical simulation, and parametric modelling of the tails of the return distribution. The latter is the recent introduction into the literature on risk management and is based on Extreme Value theory that models the tail probabilities directly without making any assumption about the distribution of the entire return process. All these methods have pros and cons, generally ranging from easy and inaccurate to difficult and precise. No method is perfect and usually the choice of a technique depends upon a market in question, and the availability of data, computational and human resources.

In this paper, we compare various approaches to the estimation of VaR of portfolios of fixed income securities supplied by the Primary Dealers Association of India (PDAI), with special focus on Extreme Value theory (EVT) and find that the EVT method provides the best VaR estimator in terms of correct failure ratio and the size of VaR. The rest of this paper is organised as follows. In section 2, we briefly discuss the advantages and disadvantages of standard approaches to VaR estimation. Section 3 presents the summary of the Extreme Value method of estimating VaR. Section 4 discusses issues related to ‘scale’ factors needed to arrive at the multi-day VaR an adequate bank capital. Section 5 outlines some specific issues related to modeling risk of a fixed income portfolio, and in particular, why we need to combine the historical simulation method with the extreme value method in computing VaR. Section 6 presents the data details and results. Section 7 concludes.

## 2 Standard approaches to VaR estimation

The most difficult part in VaR estimations is the derivation of the distribution of the underlying risk factor <sup>5</sup>. Three important stylized facts about asset returns that any risk measurement technique is expected to address are: fat-tails, volatility clustering, and asymmetry in return distribution. The fat-tail property of asset returns, while recognised long back by the researchers (Mandelbrot (1962) and Fama (1962)), has increasingly been noticed by the regulatory authorities and risk managers. “..the biggest problem we now have with the whole evolution of the risk is the fat-tailed problem, which is really creating very large conceptual difficulties. Because as we all know, the assumption of normality enables us to drop off the huge amount of complexity in our equations.. Because once you start putting in non-normality assumptions, which is unfortunately what characterizes the real world, then these issues become extremely difficult.” Greenspan (1998). To put in simple terms, the fat-tailed property would imply that one would observe extreme price movements in asset prices with a higher probability than predicted by the normal distribution. The assumption of normality for lower tail (dealing with losses) would increasingly be inaccurate, the farther into the tail that one considers the difference.

Two popular approaches in the literature for the calculation of VaR are Variance-covariance analysis and historical simulation. Variance-covariance analysis relies on the assumption that financial returns are normally distributed. This method is easy to implement because the VaR can be computed from a simple linear formula with variances and co-variances of the returns as the only inputs. Its major drawback is that the assumption that financial market returns are normally distributed is shown to be invalid in

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<sup>5</sup>There are two equivalent ways of calculating returns: change in prices expressed as a percentage, and logarithmic price difference. The latter is generally preferred for risk analysis due the analytical convenience it provides in terms of the distributions that can be specified and the link with derivative pricing.

thousands of empirical studies on asset returns (Pagan (1998)). The latter are typically characterized by fat-tails and volatility clustering. Fat-tail property of the asset returns implies that losses are much more frequent than predicted by the variance-covariance analysis. The Variance-covariance analysis becomes particularly weak where a VaR model for regulatory purposes and risk control is strong, i.e. in the prediction of extreme quantiles or large losses.

Another variant of variance-covariance analysis is the Exponential weighting approach. This approach, popularised by JP Morgan's Risk Metrics, applies exponentially declining weights to past returns to calculate conditional volatilities. This technique is justified by the presence of conditional heteroskedasticity or volatility clustering in the data, meaning that a volatile day is typically followed by volatile days. The exponential (Exp) approach also has the drawback that a conditional normality assumption needs to be made to calculate the VaR of a portfolio from its conditional volatility, an assumption that, more often than not, is not satisfied by financial data. Although the exponential smoothing approach addresses the issue of non-normality in the (unconditional) distribution of returns, it may not be applicable for the regulatory VaR purposes on three grounds. First, while daily returns reflect strong volatility clustering, they can hardly be detected in bi-weekly returns such as the regulatory 10-day holding period (Drost and Nijman (1993) and Diebold, Schuterman and Inoue (1999)). Second, the volatility clustering observed in the data largely emanates from medium and small range volatility periods. Extreme events, such as losses at or beyond 99interval scatter rather independently overtime (Jackson, Maude and Peraudin (1997) and Danielsson and de Vries (2000)). Finally, as has been established by the theoretical and empirical literature, interest rate and bond return processes typically display a complicated dependence structure in both mean and variance, thereby making the simple exponential

weighting schemes (so often used in computing the equity VaR) inapplicable (Longstaff, Schwartz and Karyoli (1998)).

The historical simulation (HS) method, by using empirical percentiles from the historical return distribution, gets around the problem of making distributional assumptions. By applying the full empirical market return distribution to all the items in the current trading portfolio, the outcome exactly reflects the historical frequency of the large losses over specific data window. Another advantage of this approach compared to variance-covariance analysis is that it can incorporate non-linear positions, such as derivative positions, in a natural way (Kupiec and O'Brien (1997)). In the context of fixed income portfolios also this property becomes very useful. We shall come back to this point later. The problem with the HS method is that it is very sensitive to the particular data window, which the Basle committee has chosen to be at least one year of past returns. In other words, whether returns from a highly volatile or a crash period is included or not makes a huge difference for the value-at-risk predicted. To put it differently, the empirical distribution function is 'dense' and smooth around the mean, so that no parametric model based on a standard distribution, such as normal, can beat the accuracy of the empirical distribution there. Due to the few occurrences of few extremely large price movements, however, it becomes 'discrete' in the tails. Hence VaR predictions based on HS exhibit high variability. Moreover, at its lower end, the empirical distribution sharply drops to zero and remains there, i.e. more severe losses in the future than the largest one during the past (sample data) is given a probability of zero, which may not clearly be the correct thing to do.

The hybrid approach proposed by Boudokh, Richardson and Whitelaw (1998) draws on the strengths of each of the above approaches to estimate the percentiles of returns directly (the HS approach), using declining weights on past observations (the Exp approach). Ordering returns over the historical

simulation period, the hybrid approach attributes exponentially diminishing weights to each observation in building the conditional empirical distribution. This would result in choice of different observations using the HS and hybrid approaches. Thus, while computation of 99 percent VaR using 850 observations involves choosing the 9<sup>th</sup> lowest observation using HS, the hybrid approach could result in the choice of a different observation. Although the BRW-HS method combines the strengths of above mentioned methods, it still suffers from the tail- discreteness problem that we discussed earlier and it may not be very effective particularly in predicting the extremely large losses.

In our view, a good value-at-risk model to satisfy regulatory minimum capital standards should correctly represent the likelihood of extreme events by providing smooth tail estimates of the portfolio return distribution which extend beyond the sample. It should also be robust to the nature of the frequency distribution followed by the asset returns. In what follows we shall briefly summarize one such method, based on Extreme Value theory, that deals with directly the modeling of the extreme tail events without making any assumptions about how asset returns are distributed.

### **3 Extreme Value theory and Value-at-Risk**

As pointed out above, financial risk management, either for the regulatory purpose or for internal control, is intimately concerned with tail quantiles (for example, the value of return,  $y$ , such that  $P(Y \geq y)=0.01$ ) and tail probabilities (for example,  $P(Y \geq y)$  for a large  $y$ ). Extreme quantiles and probabilities are of particular interest, because the ability to assess them accurately translates into the ability to manage extreme financial risks effectively. Unfortunately, the traditional parametric statistical and econometric models, typically based on estimation of entire densities, may be ill-suited to the assessment of extreme quantiles and event probabilities. Traditional

parametric models, such as variance-covariance method and Riskmetrics method, implicitly strive to produce a good fit in regions where most of the data fall, potentially at the expense of a good fit in the tails, where, by definition, few observations fall. It is common, moreover, to require estimates of quantiles and probabilities not only near the boundary of the range of observed data, but also beyond the boundary. That is, one would like to allow for the possibility that the expected loss on any future date to be greater than that observed in the past.

A key idea in the Extreme Value theory is that one can estimate extreme quantiles and probabilities by fitting a "model" to the empirical survival function (or one minus the cumulative distribution function (CDF)) of a set of data using only the extreme event data rather than all the data, thereby fitting the tail and only the tail. This approach has history in the study of catastrophic events, such as natural phenomena and disasters and the insurance field. The theory itself is developing rapidly (Embrechts et al (1997)) and there have been number notable applications in the field of finance and risk measurement (Longin (1996), McNeil and Frey (2000) and Danielsson and de Vries (2000)).

EVT uses statistical techniques that focus only that part of a sample of return data that carry information about extreme behavior. Typically, the sample is divided into  $N$  blocks of non-overlapping returns with say  $n$  returns in each block. From each block the largest rise and biggest fall in returns are extracted to create a series of maxima and minima respectively. An extreme value model (more specifically a Generalised Extreme Value (GEV) or Generalised Pareto (GP) distributions) is fitted to either of these series, via Maximum likelihood or method of moments, to estimate the 'tail index' parameter that characterizes the way the extreme events in the data can occur. Once an estimate of the 'tail index' is available, one can compute the probability of occurrence of a large event from the CDF, or VaR value

for a given probability . In this study, we estimate the tail index parameter for the lower tail (related only to the losses) using the maximum likelihood approach<sup>6</sup>.

## 4 Scale factors

### 4.1 Time aggregation factor

As has been pointed out by Kupiec and O'Brien (1995), the simple square-root- of-time formula to aggregate daily VaR estimates to 10-day VaRs need not be applicable when returns of the underlying market risk factors are non-normal. This has led to a concern among regulators whether such scaling leads to an under-estimation of the risk capital. Surprisingly, the academic literature that addressed this issue had come to an opposite conclusion that square-root-of-time rule can actually lead considerable over- estimation of the risk capital when the returns are non-normally distributed. The idea is when returns are fat-tailed (and follow any of the tail laws that we discussed earlier) extreme returns are more likely to happen than under normal distribution because the latter down weights the occurrence of large events. This might tempt the banks to use 'normal' model to estimate the VaR as the later would typically lead to a less 'inventory' of capital to be held. However, under the assumption of normality the appropriate scaling factor is square-root-of-time. On the other hand, if one takes into consideration the fat-tail property of underlying risk factors, while the estimate of VaR would typically be larger than that of the 'normal' case at a daily level, the scale factor would be  $\xi$ -root-of- time, where  $\xi$  is the tail index <sup>7</sup>. Most of the empirical estimates of tail index have found that, across countries and

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<sup>6</sup>This is a fairly non-technical overview of the EVT approach. In Appendix 1, we present a brief summary of the technical details related to the estimation of parameters of the GEV distribution of extreme returns and thereby the VaR.

<sup>7</sup>Another advantage of estimating the tail index via EV theory is that most processes are aggregable under time in their tails leading to the rule that an N-day VaR would be equal to an  $\xi$ -root-of time daily VaR.

time series,  $\xi$  is typically greater than 2, indicating that the resulting scale factor could be less than that in the case of a ‘normal’ model.

Therefore, in comparing the VaRs of a normal model vis-a-vis any fat-tailed model one faces two opposing forces: fat-tailedness while increases the daily VaR it reduces the scale factor involved in the time aggregation. Which factor dominates depends upon the degree of non-normality in the data and the time horizon over which the VaR needs to be aggregated. This, of course, is purely an empirical question. To sum up, in their enthusiasm to use a simple ‘normal’ distribution for computing multi-day VaR, banks may end up with larger capital than necessary.

## 4.2 Basle multiplication factor

The “Amendment to the Basle Capital Accord to incorporate market risks” require that banks’ 10-day, 99 percent VaR estimate has to be multiplied by a factor of at least 3 to determine the minimum regulatory capital against market risk. In addition, this factor can be increased through a variable add-on between 0 and 1 depending on the performance of banks’ 1-day VaR model in back testing procedures. The latter is expected to provide a built-in incentive for banks to develop and use better models of market risk. Stahl (1997) puts forth an interesting explanation justifying this multiple factor using a very general statistical result known as Chebychev inequality. His argument hinges on the assumption that the variance-covariance is misspecified in tails and we may not know anything about the exact return distribution, and asks what is the extreme value (associated with a given confidence level) that could cover any specification error in modeling of the market risk. At this level of generality one can, of course, justify a factor of 3 or 4. However, when one is willing to use advanced techniques to model the tail behavior of returns that reduce the specification errors significantly, it is not clear what purpose an excessively conservative multiple factor would

serve.

### 4.3 Reasons for the scale factor:

- daily VaR estimates must be translated into a capital charge that provides sufficient cushion for cumulative losses arising from adverse market conditions over an extended period of time.
- market price movements often display 'fat tails'; as a result 'normal distribution' models under estimate the risk significantly.
- past is not always a good approximation to the future.
- VaR estimates are typically based on the end-of-day positions and generally do not take into account intra-day trading risk.
- models will not be stable under exceptional market circumstances.
- models typically make simplifying assumptions to value positions in the portfolio, particularly in the case of complex instruments like options.

In the Indian context, the regulatory requirement seems to be a 30-day, 99 percent VaR with the usual Basle multiplication factor being applicable. This, in our view, is extremely conservative even by the international standards, and can act as a serious incentive for banks to look for those 1-day VaR models that give least VaR values albeit they do not match the features of the underlying risk factors. The regulator, in our view, must either stick to the international norm of 10-day VaR coupled with some multiplication factor or a 30-day VaR with far less multiplication factor or any other combination that recognizes the trade-off. We shall come back to this point when discuss the results.

## 5 Issues in modeling the risk of Fixed Income securities

The computation of VaR for a fixed income portfolio differs in important ways from that of an equity portfolio. First, unlike in the case of an equity portfolio where observed prices can be directly used for the computation of VaR, the price of each fixed income instrument in a portfolio is an outcome of many security-specific attributes, in addition to the fundamental factor, the underlying term structure. This rules out the use of prices directly for the computation of VaR if the objective is to measure the interest rate risk that the portfolio is subject to. Use of a zero coupon yield curve (ZCYC) is central to the exercise, as yield-to-maturity (YTM) based approaches are also subject to the same problem as with use of observed prices. Movements of the ZCYC, inasmuch as they depict the changes in the interest rate structure, are reflective of changes in the value of the portfolio occurring on account of interest rate changes alone.

Estimation of VaR for a portfolio of fixed income securities is complicated by two reasons: one, the changes in market values of the securities are non-linearly related to changes in spot interest rates leading to difficulties in making simple assumptions about the distribution of the portfolio returns. A related point is that since one needs to know the entire term structure of interest rates to value a fixed income security (up to the relevant maturity), to study the VaR of the security we need to model the distribution of a great number of interest rates. The popular practice of cash-flow mapping considers a selected set of interest rates and maps the cash flow timings to that of the tenor of the selected interest rates through linear interpolations. Underlying in this strategy is interest rates are distributed as normal or conditional normal, an assumption not typically supported by the data. In addition, the cash-flow interpolations may also lead to significant approximation er-

rors. A better strategy would be to generate 'returns' on (a portfolio of) fixed income instruments at the first stage by valuing the said portfolio on observed yield curves, and estimate the VaR directly from the returns on the bond portfolio. A major advantage of this approach is that it does not require an assumption about the interest rates. Since the VaR is estimated based on bond portfolio returns, this approach has the disadvantage of being portfolio specific thereby necessitating the model parametrization, and estimation to be done for each portfolio separately. The NSE-VaR system follows the latter approach of generating returns via historical simulation and fitting a model of VaR to the return series.

## **6 Data details and Results**

### **6.1 Data related issues**

The present exercise uses a hybrid approach that combines historical simulation with EV theory to estimate VaRs for a portfolio of Fixed Income securities. For the purpose of this exercise we analyse a set of 18 GSecs supplied by FIMMDA (see Table 1 in Appendix 1). Although the VaRs are computed both for the individual securities as well as the portfolio, we believe that one should exercise caution in using the former for the capital adequacy purposes for two reasons. First, from works of Altzner, Delbaen, Elber and Heath (1998), we know that the VaR as a measure of risk is not sub-additive in the sense that the sum of individual securities' VaRs could be either greater or less than the portfolio VaR. This is contrary to the general belief that the former is always greater than the latter, thereby providing a conservative estimate of portfolio VaR. This belief is generally valid only for normally distributed returns. As a result, knowing security specific VaRs could at best be indicative of the contributing factors to overall risk and not useful for setting the exact level of capital to match the

market risk <sup>8</sup>. Second, it may be tempting to compare the security specific VaR with the standard measures of interest risk such as duration etc. This is definitely not correct as the concept of duration as a measure of risk has valid interpretation only within the framework of YTM (and not ZCYC) and, even in the YTM framework, it does not say anything about the probability of observing a unit change in interest rates. VaR, on the other hand, is a combined statement about how value of bond changes in response to a unit change in interest rates and interest rates are expected to change for a given level of probability. For the purpose of this exercise we use the data base of daily Zero Coupon Yield Curves that we have developed at NSE for the period 1998 February to April 2001. In estimation of VaR (for any method) an important choice is related to the size of in sample and out-of sample of observations. The former is used in the estimation of the model while the later is used for testing. In our view, the in sample data must be long enough to cover maximum possible variation in the data. That is if one estimates a model based on a very ‘calm’ period, it can never predict (under any model) well during volatile periods. To guard against such biases, we use 850 data points in our database (approximately covering a period of three years) that include both volatile and tranquil periods. Also, a large in-sample data would increase the precision with which can estimate parameters of various models. For the purpose of back testing, we use about 250 observations (approximately last 1 year of data). The reason for including a 1-year period is that to test for a 99 percent VaR model’s performance against the actual data, we need at least 100 observations in the test data set.

The estimation is done, first, on a sample of 1 to 850 observations with the estimated VaR for a security (portfolio) is compared with the 851<sup>st</sup>

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<sup>8</sup>The Expected Shortfall (ES) measure developed in Altzner et all (1998), on the contrary, is coherent in the sense that it is strictly sub-additive. We have computed ES as well for each security in the said portfolio.

actual data. Then the estimation is done on the sample 2 to 851 with the estimated VaR being compared with the 852<sup>nd</sup> actual data point, and so on. The performance criteria used for comparing models are: number of failures (out of total 250 test cases) and the average VaR value (as percentage in total value of the security/portfolio). This analysis is done at 1-day holding period. All VaRs are computed at 99 percent level of confidence. Intuitively, we would expect that if any model characterizes the data correctly, the number of failures (i.e. the actual value is less than the negative of VaR number) should be 3. If this number is less (grater) than two, the model is too conservative (liberal). As pointed out by Kupiec and O'Brien (1997), the confidence interval for the failure ratios is 2 to 5. Therefore, we can consider a model being too liberal in the sense of having a lower VaR number only if the number of failures is greater than 5.

For multi-day forecasts, one could have conducted the entire analysis at multi-day horizons directly i.e. model the multi-day non-overlapping returns and compare them with the actual multi-day returns. However, this approach can not be taken in our context as the time series of ZCYCs needed for portfolio return computation is too short to give sufficient number of non-overlapping multi-day return sample. Modeling the multi-day overlapping returns is fraught with serious problems. In an earlier exercise, we observed that such a return series displays a very complicated structure of dependence in the data, and as a result estimates of VaR are highly imprecise and unstable. Therefore, we will have to estimate various models on 1-day returns on each security (portfolio), aggregate them to the relevant time horizon and compare with its actual multi-day change in value in the test data set. In computing portfolio returns, the weights assigned to each security are equal to their outstanding issue size as a proportion of total outstanding issue of all these securities.

The estimation of VaR under variance-covariance and HS methods is

straight- forward and well understood. The EV approach, on the other hand involves a number of steps. The first step entails the choice of block length, which determines the number of returns in the base data from which we choose maxima (minima). Clearly the number of maxima / minima return series created in this manner depends on the choice of the block length and the total sample size. Christoffersen et al (1998), and Longin (1998) suggest that block lengths of size 10 to 25 (within a total sample of at least 1000) produce relatively precise and robust estimates of the tail index. In this study we use block lengths of 5, 10, 25; the results, in terms of tail parameters, are qualitatively similar. Hence we report the results based on the block length of 10. The tail index parameter (along with the scale and location parameters that characterize a standardized extreme value) is estimated using the maximum likelihood approach. All estimations are done using the Constrained Optimization (CO) module of GAUSS<sup>9</sup>.

## 6.2 Results

In our empirical exercise we compute the VaRs of each and portfolios of fixed income instruments supplied to us by the Primary Dealers' Association of India. In all we analyse ten live PDAI portfolios as of end of September, 2001. For the reasons of confidentiality we do not disclose the names of them in reporting the results.

As a first step in estimating various VaR models, we examine the time series properties of each security and portfolio returns. The results (not reported) indicate that, the returns are highly non-normal with significantly positive skewness and excess kurtosis. There is also an evidence of a significant dependence both in mean and variance <sup>10</sup>. These results, in our view,

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<sup>9</sup>Results on parameter estimates and GAUSS code are available upon request

<sup>10</sup>When the return series are fitted with AR(p)-GARCH(1,1) model, the GARCH models parameters typically have explosive roots (i.e. sum of the slope coefficients is significantly greater than one) indicating the inapplicability of standard conditional heteroskedastic models. This result is consistent with some other studies in the literature related to US

question the (unconditional / conditional) normality assumption that the variance-covariance and Riskmetrics approaches regularly make.

We next compare the estimates of VaR for each security and the total portfolio across three methods - Normal, HS, and EVT - for three horizons - 1-day, 10-day and 1- month. Tables 1 to 3 in the Appendix 2 report these results for 1-day, 10-day and 1-month horizons respectively. We report the number of model failures and an average VaR over 250 test cases.

**The principal findings are:**

- At a one-day horizon (Table 1), all of the three methods under consideration give more than theoretically correct number of failures i.e 2. However, the number of failures are within the confidence interval and hence one can not choose one model over the other as they are not statistically different from each other based on the number of failures alone. The average VaR, expressed as a percentage of the market value, differs significantly across the methods. Typically, the average VaR for EVT is less than that for Normal and HS methods across securities and portfolios. The apparently counter-intuitive result the VaR implied by EVT less than Normal may be because when one models VaR as a function of only the unconditional variance (as in the case of normal method), neglecting the conditional dependencies in the data, an upward bias in the estimated variance (and VaR) is expected to result. The extremal observations, on the other hand, being robust to dependencies in the center of the distribution, are not expected to be affected. These results imply that at 1-day horizon, while we can not choose any model based on failure events, the average VaR implied by EVT is significantly lower than other methods. The latter fact can not, of course, be over-emphasized as in principle EVT VaR could as

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well have been greater than the other two.

- At 10-day and 1-month horizons (Tables 2 and 3 respectively), the VaR estimates of Normal and HS method (with square-root-of-time scaling) leads to excessively conservative estimates as reflected in 0 failure events across all securities and portfolios. The EVT method, on the other hand, leads to number of failures closer to the theoretical value though being conservative. More significantly, the gains in terms a lower VaR are now substantial in the sense that at 10-day horizon the average VaR (over portfolios) implied by EVT is less than that of normal by about 60 percent and than HS by about 90 percent. At 1-month horizon these differences are even more significant at about 80 and 150 percent respectively. Clearly the square-root-of-time rule seem to generate excessively large VaR numbers both in the case of normal and HS methods. Considering that banks are expected to make a provision for capital at least 3 time the estimated VaR, the differences in these methods further gets magnified. We interpret these results to imply that the time-aggregation rules associated with simple models might lead to excessively large VaR numbers, and EVT would provide more accurate estimate of the latter. As mentioned above, the fact that the average VaR numbers generated by EVT being significantly less than the other methods could be specific to the portfolios and sample period analysed. The merit of using EVT for measuring VaR should be on the grounds of conceptual soundness in that the latter accounts for the fat-tail phenomenon in the data more accurately along with a correct time- aggregation factor.
- Although we have not under taken a thorough analysis of the role Basle multiplication factor, we believe that if the extreme tails are modeled appropriately, using say EVT, a large multiplication factor

of 3 or 4 is appears excessively conservative in the sense that never in the entire history of data one finds a loss that comes any where near such a value in the case of 10-day horizon. This is more so for the 1-month horizon. Therefore, we suggest that the regulator must either stick to the international norm of 10-day VaR coupled with some multiplication factor or a 1-month VaR with far less multiplication factor or some other combination. A thorough analysis to study this aspect on live portfolios of PDs would be highly instructive. In tune with the Basle's internal models approach to create incentives for banks to adopt better risk measurement and management techniques, the regulator may insist on an add-on factor that increases with the failures of the internal VaR model at 1-day horizon.

## 7 Conclusion

In this exercise, we have presented a case for a new method of computing the VaR for a set of fixed income securities based on extreme value theory that models the tail probabilities directly without making any assumption about the distribution of entire return process. We also compare the estimates of VaR for a portfolio of fixed income securities across three methods: Variance-Covariance method (under the assumption that returns on fixed income instruments are normally distributed), Historical Simulation method and Extreme Value method. We find that extreme value method provides the best VaR estimator in terms of correct failure ratio and lowest VaR for the representative sample of portfolios of ten PDs. For the purpose of risk based bank capital regulation, it is instructive to extend this analysis over all PDs and for various scale factors.

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## Appendix 1 - Extreme Value Theory - an overview <sup>11</sup>

Extreme value theory is a branch of the theory of order statistics, which dates back to the pioneering works by Frchet (1927), and Fisher and Tippet (1928), and the celebrated extremal type theorem of Gnedenko (1943). Gumbel (1958) gives a clear presentation of the important elements of the theory and more recent and advanced treatments can be found in the references cited earlier. In what follows we briefly summarize the basic theory used in the present study.

Let the basic return observed on the time interval  $[t-1,t]$  of length  $f$  is denoted by  $R_t$ . Let us call  $F_R$  the cumulative distribution function of  $R$ . It can take values in the interval  $(1,u)$ . For a normally distributed variable, the limits range from  $-\infty$  to  $+\infty$ . Let  $R_1, R_2, \dots, R_n$ , be the returns observed over  $n$  basic time intervals  $[0,1], [1,2], \dots, [T-1,T]$ . For a given return frequency  $f$ , the two parameters  $T$  and  $n$  are linked by the relation  $T=nf$ . Extremes are defined as the minimum and maximum of  $n$  random variables  $R_1, R_2, \dots, R_n$ . Let  $Z_n$  denote the minimum observed over  $n$  trading intervals:  $Z_n = \text{Min}(R_1, R_2, \dots, R_n)$  <sup>12</sup>. Assuming that returns  $R_t$  are drawn from an i.i.d  $F_R$ , the exact distribution of the minimal return is given by:

$$F_{Z_n}(z) = 1 - (1 - F_R(z))^n \quad (1)$$

The probability of observing a minimal return above a given threshold is denoted by  $p^{ext}$ . This probability implicitly depends upon the number of basic returns  $n$  from which the minimal return is selected. The probability of observing a return above the same threshold over one trading period is

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<sup>11</sup>This section draws heavily from Longin (2000) and McNeil (1999).

<sup>12</sup>We focus on the modelling the minima as we are interested in measuring losses of a long-position. But, all the results presented in this section will carry over for analysing maxima series by transforming the random variable  $R$  into  $-R$  by which the minimum becomes maximum and vice versa. Analysis of the maxima (upper tail) would be of use in studying the downside risk of a short position.

denoted by  $p$ . From Eq. (1) the two probabilities,  $p^{ext}$  and  $p$ , are related by the equation:  $p^{ext} = p^n$ .

In practice the distribution of returns is not precisely known and, therefore, neither is the exact distribution of the minimal returns. From Eq. (1), it can also be concluded that the limiting distribution of  $Z_n$  obtained by letting  $n$  tends to infinity is degenerate in the sense that it is degenerate for  $z$  less than the lower bound  $l$ , and equal to 1 for  $z$  greater than  $l$ .

To find a non-degenerate limiting distribution, the minimum  $Z_n$  is normalised with a scale parameter  $\alpha_n$ , and location parameter  $\beta_n$  such that the distribution of the standardised minimum  $(Z_n - \beta_n)/\alpha_n$  is non-degenerate. The so-called **extreme value theorem** specifies the form of the limiting distribution as the length of the time-period over which the minimum is selected (the variables  $n$  or  $T$  for given frequency  $f$ ) tends to infinity. The limiting distribution of the minimal return, denoted by  $F_Z$ , is given by

$$F_Z(z) = \begin{cases} 1 - \exp(-(1 + \xi z)^{-1/\xi}) & \text{if } \xi \neq 0, \\ 1 - \exp(-e^{-z}) & \text{if } \xi = 0 \end{cases} \quad (2)$$

for  $(1 + \xi z) > 0$ . The parameter  $\xi$ , called the tail index, models the distribution tail. Feller(1971) shows that the tail index value is independent of the the frequency  $f$  implying that the tail is stable under time-aggregation. According to the tail index value three types of distributions are distinguished: the Frechet distribution ( $\xi > 0$ ), the Weibull distribution ( $\xi < 0$ ) and the Gumbel distribution ( $\xi = 0$ ).

The Frechet distribution is obtained for fat-tailed distributions of returns such as Student and stable Paretian distributions. The fatness of the tail is directly to the tail index. More precisely, the shape parameter  $k$  (equal to  $1/\xi$ ) represents the maximal order of finite moments. For example, if  $k$  is greater than 1, mean of the distribution exists;  $k$  is greater than 2, variance

is finite and so on. The shape parameter is an intrinsic parameter of the distribution and does not depend on the number of returns  $n$  from which the minimal returns are selected. The shape parameter corresponds to the number of degrees of freedom of Student distribution and to the characteristic exponent of a stable Paretian distribution.

The Weibull distribution is obtained when the distribution of returns has no tail (we cannot observe any observations beyond a given threshold defined by the end point of the distribution). Finally, the Gumbel distribution is reached for thin-tailed distributions such as the normal or log-normal distributions. For small values of  $\xi$  Fréchet and Weibull distributions are very close to Gumbel distribution.

These theoretical results show the generality of the extreme value theorem: all the mentioned distributions of returns lead to the same form of distribution for the extreme return, the extreme value distributions obtained from different distributions of returns being differentiated only by the value of the scale and location parameters and the tail index.

### **EV theory for conditional processes**

Till now we have presented the result under the assumption that the basic return series are i.i.d, an assumption generally not satisfied. However, it can be shown that a great deal of results could be carried over, with modifications, for stationary random sequences. For processes whose dependence structure is not "too strong" (such as volatility clustering), the same limiting extreme-value distribution  $F_Z$  given by Eq. (2) is obtained (Leadbetter et al. (1983)). With stronger dependence structure the behaviour of extremes is affected by the local dependence in the process as clusters of extreme values appear. In this case it can still be shown that an extreme value modelling can be applied, the limiting extreme-value distribution being equal to

$$F_Z(z) = 1 - \exp(-(1 + \xi z)^{-\theta/\xi}) \quad (3)$$

where the parameter  $\theta$ , called the extremal index, models the relationship between the dependence structure and the extremal behavior of the process. This parameter is related to the mean size of clusters of extremes (see Embrechts et al (1997) and McNeil (1998)). The extremal index  $\theta$  verifies:  $0 \leq \theta \leq 1$ . The equality  $\theta = 1$  is obtained in the case of weak dependence and independence. In other cases, the stronger the dependence, the lower the extremal index. For a given block size  $n$ , number of blocks  $m$ , and a high threshold  $u$ , the asymptotic estimator of  $\theta$  is

$$\theta = n^{-1} \frac{\ln(1 - (K_u/m))}{\ln(1 - (N_u/mn))} \quad (4)$$

where  $N_u$  is the number of exceedances of the threshold and  $K_u$  is the number of blocks in which the threshold is exceeded. For  $K_u/m$  and  $N_u/mn$  small, this estimator reduces to  $K_u/N_u$ .

Berman (1964) shows that the same form for the limiting extreme-value distribution is obtained for stationary normal sequences under weak assumptions on the correlation structure. Leadbetter et al. (1983) consider various processes such as discrete mixture of normal distributions and mixed diffusion jump processes all have thin tails so that they lead to a Gumbel distribution for the extremes. Longin (1997) shows that the volatility of the process returns (modelled by the class of ARCH processes) is mainly influenced by the extremes. De Haan et al. (1989) show that if returns follow an ARCH process, then the minimum has a Fréchet distribution.

### **Steps involved in the computation of VaR**

**step 1: Choose the frequency of returns.** The choice of the frequency should be related to the degree of liquidity and risk of the position. For a liquid position, high frequency returns such as daily returns can be selected as the assets can be sold rapidly in good market conditions. The frequency should be high as extreme price changes in financial markets tend to occur during very short-time periods (Kindelberger (1978)). Low frequency returns may not be relevant for a liquid position as the risk profile could change very rapidly. For an illiquid position, low-frequency returns like weekly or monthly return could be a better choice since the time to liquidate the assets in good market conditions may be longer. However, the choice of a low frequency implies a limited number of extreme observations, which could impact adversely upon the analysis as extreme value theory is asymptotic by nature. The problem of infrequent trading may be dealt with some data adjustment as done in Lo and McKinlay (1990) and Stoll and Whalley (1990). The choice of frequency may also be guided or imposed by regulators. For example, the Basle Committee (1996) recommends a holding period of 10 days.

**step 2: Build the history of returns of the position  $R_t$ .** Based on the time-series of the underlying risk factors, returns on a position should be computed.

**step 3: Choose the length of the period of selection of minimal returns.** The selection period should consist of long history of returns computed over non-overlapping periods with frequency  $f$ . As the extreme value theory is essentially asymptotic in nature, extreme returns have to be selected over time-periods long enough that the exact distribution of minimal returns can be safely replaced by the asymptotic distribution. An implication of this constraint is that one may have to stick to high-frequency data as there may not be enough number of non-overlapping data points at lower frequency. When the focus of analysis is multi-day VaR, then one has to

derive the latter from a daily VaR with appropriate time-aggregation rule.

**step 4: Choose the block length.** The choice of block length, for a given number of observations in the database would determine the number of minimal returns to be used in the tail estimation. That is the period covered by the database is divided into non-overlapping sub-periods each containing  $n$  observations of returns of frequency  $f$ . For each sub-period, the minimal return is selected. From the first set of  $n$  observations of basic returns  $R_1, R_2, \dots, R_n$ , one takes the lowest observation denoted by  $Z_{n,1}$ . From the next  $n$  observations  $R_{n+1}, R_{n+2}, \dots, R_{2n}$ , another minimum called  $Z_{n,2}$  is taken. From  $nN$  observations of returns, a time series  $(Z_{n,i})_{i=1,N}$ , containing  $N$  observations of minimal returns is obtained.

**step 5: Estimate the parameters of the GEV distribution.** Three parameters  $\alpha_n$ ,  $\beta_n$  and  $\xi$  of the asymptotic distribution  $F_{Z_n}$  is estimated using the maximum likelihood method. The ML estimate provides asymptotically unbiased and minimum variance estimates. Note also that the maximum likelihood can also be used for the three types of extreme value distribution (Frechet, Weibull, and Gumbel) while other estimators such as the method of moments estimator developed by Hill (1975) are valid for the Frechet case only. The extremal index  $\theta$  could be estimated prior to the ML estimation using the 'blocks' method suggested in Embrechts et al. (1997) and McNeil (1998).

**step 6: Choose the value of the probability  $p^{ext}$ .** In the extreme value method, instead of the probability related to a basic return, the probability related to a minimal return is used: for example, the probability of a minimal daily return observed over a 'block' being above a given threshold. As explained above, for an independent or weakly dependent processes,  $p^{ext} = p^n$ . Under the latter condition,  $VaR(F_{Z_n}, p^n) = VaR(F_R, p)$ . The choice of the probability  $p^{ext}$  is arbitrary as in other methods. Basle committee specifies a value of 99 percent for  $p$  implying a value for probability

$p^{ext}$  equal to  $(0.99)^n$  assuming weak dependence. For processes presenting strong dependence, the probability  $p^{ext}$  would be equal to  $((0.99)^n)^\theta$ , where  $\theta$  is the extremal index.

**step 7: Compute the VaR.** The last step consists of computing the VaR of a position for a given estimate of GEV parameter vector, and the probability  $p^{ext}$ . The VaR could be derived from the following probability relation for GEV distribution:

$$p^{ext} = (1 - F_Z^{asym}(-VaR)) = \exp[-(1 + \xi(-VaR - \beta_n)/\alpha_n)^{-\theta/\xi}] \quad (5)$$

leading to

$$VaR = -\beta_n + (\alpha_n/\xi)[1 - (-\ln(p^{ext}))^{-\xi/\theta}] \quad (6)$$

## Appendix 2 - Results Tables

**Table 1** Portfolio-wise estimates of 99 percent VaR - 1-day

Security	Number of Failures				Average VaR (percent)				Duration
	Normal	HS	WHS	EVT	Normal	HS	WHS	EVT	Dur.
<b>PD1</b>	4	3	5	3	0.45	0.62	0.56	0.52	1.95
<b>PD2</b>	4	3	6	4	0.63	0.79	0.54	0.69	3.69
<b>PD3</b>	4	3	7	4	0.71	0.86	0.52	0.77	4.19
<b>PD4</b>	4	3	6	3	0.77	0.96	0.49	0.86	4.67
<b>PD5</b>	3	3	5	3	0.85	1.07	0.58	0.99	5.04
<b>PD6</b>	3	3	8	3	0.86	1.15	0.63	0.92	5.12
<b>PD7</b>	4	2	5	3	0.87	1.15	0.53	1.01	5.24
<b>PD8</b>	4	2	5	4	1.02	1.3	0.68	1.14	5.87
<b>PD9</b>	4	2	6	4	1.02	1.34	0.65	1.15	6.04
<b>PD10</b>	3	2	4	3	1.36	1.79	0.88	1.56	7.78

**Table 2** Portfolio-wise estimates of 99 percent VaR - 10-day and 1 month

Security	10-Day (percent)				1-Month (percent)			
	Normal	HS	WHS	EVT	Normal	HS	WHS	EVT
<b>PD1</b>	1.42	1.96	1.77	1.06	2.46	3.40	3.07	1.49
<b>PD2</b>	1.99	2.50	1.71	1.48	3.45	4.33	2.96	2.12
<b>PD3</b>	2.25	2.72	1.64	1.72	3.89	4.71	2.85	2.53
<b>PD4</b>	2.43	3.04	1.55	2.16	4.22	5.26	2.68	3.35
<b>PD5</b>	2.69	3.38	1.83	1.89	4.66	5.86	3.18	2.57
<b>PD6</b>	2.72	3.64	1.99	2.59	4.71	6.30	3.45	4.25
<b>PD7</b>	2.75	3.64	1.68	2.54	4.77	6.30	2.90	3.94
<b>PD8</b>	3.23	4.11	2.15	2.93	5.59	7.12	3.72	4.60
<b>PD9</b>	3.23	4.24	2.06	2.96	5.59	7.34	3.56	4.64
<b>PD10</b>	4.30	5.66	2.78	3.83	7.45	9.80	4.82	5.88